Euclidean Medial Axis computation

David Coeurjolly

Laboratoire LIRIS
Université Claude Bernard Lyon 1
43 Bd du 11 Novembre 1918
69622 Villeurbanne CEDEX
France
david.coeurjolly@liris.cnrs.fr

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Table of contents

1. Definitions
2. MA with Chamfer metrics
3. MA with the Euclidean metric
4. Applications
5. Conclusion
# Table of contents

1. Definitions
2. MA with Chamfer metrics
3. MA with the Euclidean metric
4. Applications
5. Conclusion
Skeleton and medial axis in the continuous plane

Possible definitions

- generalized symmetry axes
- *prairie fire model* and “self-intersections” of wave-fronts
- centers of maximal balls
- one dimensional topological equivalent
- ...

All these definitions are equivalent in the continuous case

In the discrete model

- Skeleton: *minimal* topological equivalent
- Medial axis: centers of maximal balls
Skeleton and medial axis in the continuous plane

Possible definitions

- generalized symmetry axes
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- …

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In the discrete model

**Skeleton**: *minimal* topological equivalent

**Medial axis**: centers of maximal balls
Link

\( DT(p) \) : maximal radius such that the disk centered at \( p \) with radius \( DT(p) \) is included in the shape

\( \Rightarrow \) the medial axis can be viewed as the set of local maxima in the DT
Maximal disks and Medial axis

**Definition (Maximal ball)**

A maximal ball is a ball contained in the shape not exactly covered by another ball contained in the shape.

**Definition (Medial axis)**

The medial axis (MA for short) of a shape is the set of maximal ball centers contained in the shape.
Table of contents

1 Definitions
2 MA with Chamfer metrics
3 MA with the Euclidean metric
4 Applications
5 Conclusion
Let \( p \) be a point in the shape, how to decide if \( p \in MA \)?

Look-up table \([DT(p), \vec{d}_i] \rightarrow [\text{limit radius}]\)

\[ p \in MA \iff \forall i, DT(p + \vec{d}_i) < LUT_i(p) \]
MA extraction with Chamfer metrics

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\[ a \]
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p \in MA \iff \forall i, \ DT(p + \vec{d}_i) < LUT_i(p)$
Example (d_{3,4})

\( p \in MA \iff \forall i, DT(p + \vec{d}_i) < LUT_i(p) \) with \( \{a, b\} = \{\rightarrow, \uparrow\} \)

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Example \((d_{3,4})\)

\[ p \in MA \iff \forall i, DT(p + \vec{d}_i) < LUT_i(p) \text{ with } \{a, b\} = \{\rightarrow, \nearrow\} \]

\[
\begin{array}{c|cc}
DT(p) & a & b \\
\hline
3 & 4 & 5 \\
4 & 7 & 8 \\
6 & 8 & 9 \\
7 & 10 & 11 \\
8 & 11 & 12 \\
9 & 12 & 13 \\
10 & 13 & 14 \\
11 & 14 & 15 \\
12 & 15 & 16 \\
13 & 16 & 17 \\
\end{array}
\]
Example $(d_{3,4})$

$p \in MA \iff \forall i, DT(p + \bar{d}_i) < LUT_i(p)$ with \(\{a, b\} = \{\rightarrow, \nearrow\}\)

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Example diagram:

- Nodes 3, 4, 6 connected with arrows indicating the direction of the Chamfer metrics.
- The example illustrates the calculation of $DT(p + \bar{d}_i)$ for $p \in MA$.
MA with Chamfer metrics - summary

:-)  
- Local computations to detect maximal balls based on a LUT
- Can be generalized to higher dimensions

:-(
- Anisotropic representation
- The entire LUT must be computed
- A new LUT is required if you change the chamfer mask (weights or displacements)
Definitions

MA with Chamfer metrics

MA with the Euclidean metric

Applications

Conclusion
## Medial axis for the Euclidean metric

**LUT**

We can also construct a Look-up table but the number of possible displacements is unbounded.

**Separable techniques**

Optimal algorithms to solve:

- the reversible Euclidean distance transform (REDT)
- the MA extraction problem (using Laguerre Diagram)
Problem

Given a MA (i.e. a set of disks), how to reconstruct the shape?

Let \((x_m, y_m, r_{x_m}y_m)\) be the points of the MA. The object \(P\) is given by:

\[
P = \{(i, j) | \exists m, (i - x_m)^2 + (j - y_m)^2 < r_{x_m}y_m \}\.
\]

or

\[
P = \{(i, j) | \max_{(x_m, y_m) \in MA} \{r_{x_m}y_m - (i - x_m)^2 - (j - y_m)^2\} > 0 \}.
\]

⇒ Same kind of maximization/minimization processes as in the SEDT

SEDT:

\[
s(q) = \min_{p(x, y) \in P} \{(x - i)^2 + (y - j)^2\}
\]
The REDT

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SEDT:

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Separable process

\[ P = \{(i, j) \mid \max_{(x_m, y_m) \in MA} \{r_{x_m y_m} - (i - x_m)^2 - (j - y_m)^2\} > 0\} . \]

Let \( F \) be the image where \( f_{ij} = r_{ij} \) if \((i, j, r_{ij}) \in MA\) or 0 otherwise

1. Build from \( F \) the map \( G = \{g_{ij}\} \) such that:
   \[
g_{ij} = \max_x \{f_{xj} - (i - x)^2, 1 \leq x \leq n\} .
   \]

2. Build from \( G \) the map \( H \) such that:
   \[
h_{ij} = \max_y \{g_{iy} - (j - y)^2, 1 \leq y \leq n\} .
   \]
Upper envelope computation

\{ f_{xj} - (i - x)^2 \}: family of parabolas

![Graph showing upper envelope computation with equations and tables]
REDT Computational Cost

Upper envelope computation in linear time

$\Rightarrow O(n^2)$ for a 2-D image
$\Rightarrow O(n^d)$ for a d-D image
Definitions

- 1 disk of the MA = 1 elliptic paraboloid \( r_{x_m y_m} - (i - x_m)^2 - (j - y_m)^2 \)
- Compute the \( \max \) function ⇔ Compute the upper envelope of all the elliptic paraboloids

⇒ Laguerre Diagram
Power diagrams / Laguerre diagrams

\[ P = \{(i, j) \mid \max_{(x_m, y_m) \in MA} \{r_{x_m y_m} - (i - x_m)^2 - (j - y_m)^2\} > 0\}. \]

**Definitions**

The power of a point with respect to a circle \((i, j, s_{ij})\):

\[ \sigma(x, y) = (x - i)^2 + (y - j)^2 - s_{ij}^2 \]

Laguerre Cell : \(L(\sigma_i) = \{x \in \mathbb{R}^2 \mid \sigma_i(x) \leq \sigma_j(x), 1 \leq j \leq n\}\)

Laguerre Diagram : the set of non-empty cells with their faces
Links between MA and Laguerre Diagram

Properties

- Associate to the point \((i, j)\) the maximal elliptic paraboloid at \((i, j)\) = Find the Laguerre cell in which \((i, j)\) belongs to

- Equivalence between maximal disks and maximal elliptic paraboloids

⇒ Extract MA = find the sites of the non-empty cells in the Laguerre Diagram
Observation

If we have two disks $B_1$ and $B_2$ contained in the shape such that $B_1 \subset B_2$, and if we mark the disks that belong to the upper envelope of elliptic paraboloids in the REDT algorithm, $B_1$ will not be marked.

Ideas of the algorithm

- Input: all disks contained in the shape, i.e. $\{(x, y, r_{xy})\}$ for all $(x, y) \in S$ and $r_{xy} = SDT(x, y)$
- Use the REDT algorithm and the upper envelope computation to filter the disks to obtain the MA
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REDT $\rightarrow$ Discrete Laguerre Labeling

Sketch of the alg.

- For each point $(i, j)$ of the object, we associate to $(i, j)$ the label to the parabola in the the upper envelope that is maximal in $(i, j)$
- List of upper envelope parabolas are stored in $s[q]$
- Labeling performed dimension by dimension

Computational cost

$\Rightarrow O(n^2)$ for a 2-D image
$\Rightarrow O(n^d)$ for a d-D image
Discrete Laguerre Diagram → MA centers

Filtering of the Laguerre Diagram

\[ D \text{ and } E \text{ belong to the upper envelope but the discrete disk } D \text{ contains the discrete disk } E \]

⇒ Linear in time filtering
Summary

EDT

Voronoi

Rewriting

MA

Laguerre

Non-empty cells
MA filtering in computational geometry

Observations

- small disks may not be essential for the shape geometry
- contact points between the disk and the object contour are important

Filtering based on two parameters [D. Attali]

1. Radius of the disk
2. Contact angle (bisector angle)
MA filtering in computational geometry

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**Filtering based on two parameters [D. Attali]**

1. Radius of the disk
2. Contact angle (*bisector angle*)

![Diagram showing filtering based on radius and contact angle](image)
MA filtering results [D. Attali]
MA filtering results [D. Attali]
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Shape coding and Transmission

Idea

- Use MA balls to efficiently lossless encode binary shapes
- Use convex hull of 2 balls (*a.k.a.* cone)
- Use convex hull of n balls, ...
Use MA balls to estimate the normal vectors and to smooth surfaces during ray-tracing

[S. Prévost and L. Lucas]
Table of contents

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We have:

- optimal in time and separable algorithms to compute the REDT and the MA;
- Algorithms based on the error free Euclidean metric;
- deep links between Discrete Geometry and Computational Geometry.