2nd talk: Minimal path-costs
Computation and applications

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Outline

• What is a path cost?
• Methods
  − Geodesic distance
  − Fuzzy connectedness
  − Fuzzy distance
  − Minimum barrier distance
• Algorithms for efficient computation
• Application - segmentation
Digital images, paths

4-connected paths

Topographic representation
Path cost

Intensity profile

Cost function: curve length

Path cost

Intensity profile

Cost function: curve integral (area under curve)

Cost function: max value
Path cost

Intensity profile

Cost function: min value

Cost function: minimum barrier (difference, max and min)
Optimal/minimal cost paths

Methods

- Curve length - Geodesic distance
- Max/min value - Fuzzy connectedness
- Curve integral - Fuzzy distance
- Minimum barrier distance
**Methods**

Node weighted graph

Image, intensity values

Graph representation


Edge weighted graph
Curve length - Geodesic distance

- Geodesic distance – Curve length on topographic image representation
- Cost: Sum of edge weights
- Simple $f(m, n) = |m - n| + 1$
  (or $f(m, n) = \sqrt{(m - n)^2 + 1}$)
- Example path cost:
  $f(n_{1,1}, n_{2,1}) + f(n_{1,1}, n_{2,2}) + f(n_{2,2}, n_{2,3}) = 3 + |n_{1,1} - n_{2,1}| + |n_{2,1} - n_{2,2}| + |n_{2,2} - n_{2,3}|$

Geodesic distance

Extensions of the simple cost function:
- Distance Transform on Curved Spaces (DTOCS)
  - Approximation of local cost
- Weighted DTOCS (WDTOCS)
  - Approximation of local distances

Fuzzy Connectedness

Fuzzy set, fuzzy membership functions:
A fuzzy set $S$ in a fuzzy space $X$ (here $X \subset \mathbb{Z}^2$) is a set of ordered pairs $S=\{(x,\mu_S(x))|x \in X\}$, where the membership function $\mu_S:X \rightarrow [0,1]$ represents the grade of membership of $x$ in $S$.

Max/min value - Fuzzy Connectedness

- $f$ defined by fuzzy affinity
- Cost: minimum of edge weights
- Simple $f(m,n) = 1 - |m-n|
- Example path cost:
  $\min \left( f(n_{1,1},n_{2,1}), f(n_{2,1},n_{2,2}), f(n_{2,2},n_{2,3}) \right) = 1 - \max(|n_{1,1} - n_{2,1}|, |n_{2,1} - n_{2,2}|, |n_{2,2} - n_{2,3}|)$
- Gives fuzzy membership values
Fuzzy Connectedness

Extension of the simple cost: support for prior information:

- Homogeneity-based component: The degree of local hanging-togetherness due to the similarity in intensity
- Object-feature based component: The degree of local hanging-togetherness with respect to some given feature

A. Rosenfeld, “The fuzzy geometry of image subsets”, Pattern Recognition Letters, 2(5), 1984

Fuzzy Connectedness

Extensions/variants

- Local scale
- Vectorial fuzzy connectedness
- Relative fuzzy connectedness
- Iterative relative fuzzy connectedness
- …
Curve integral

- Cost: Sum of edge weights
- Simple $f(m, n) = \frac{|m+n|}{2}$
- Example path cost:
  $f(n_{1,1}, n_{2,1}) + f(n_{2,1}, n_{2,2}) + f(n_{2,2}, n_{2,3}) = \frac{n_{1,1}+2n_{2,1}+2n_{2,2}+n_{2,3}}{2}$

- Stability to noise and blur
- Generalization of path costs (chessboard/city block distances) for binary images
- Fuzzy distance - Curve integral typically applied to, for example, (inverted) fuzzy membership values or edge magnitude images

D. Rutovitz “Data structures for operations on digital images” Pictorial Pattern Recognition, Washington, Thompson 1968
Minimum Barrier Distance

• \( f \) vector valued function
• Cost: minimum interval
  \[ f(m, n) = (\min(m, n), \max(m, n)) \]

• Example path cost:
  \[ |\min(f_1(n_{1,1}, n_{2,1}), f_1(n_{2,1}, n_{2,2}), f_1(n_{2,2}, n_{2,3})) - \max(f_2(n_{1,1}, n_{2,1}), f_2(n_{2,1}, n_{2,2}), f_2(n_{2,2}, n_{2,3}))| = \max(n_{1,1}, n_{2,1}, n_{2,2}, n_{2,3}) - \min(n_{1,1}, n_{2,1}, n_{2,2}, n_{2,3}) \]

Minimum barrier distance

• Efficient algorithms
• Metricity
• Convergence properties
• Vectorial minimum barrier distance

**Minimum barrier distance property**

Let Π be the set of all paths between two points p and q in $\mathbb{Z}^n$ and $I: \mathbb{Z}^n \rightarrow \mathbb{R}$ (the image).

The minimum barrier distance between the points is

$$\min_{\pi \in \Pi} \left( \max_t I(\pi(t)) - \min_t I(\pi(t)) \right)$$

Let $\Pi'$ be the set of all paths between points p and q in $\mathbb{R}^n$ and $I: \mathbb{R}^n \rightarrow \mathbb{R}$, then

$$\inf_{\pi \in \Pi'} \left( \max_t I(\pi(t)) - \min_t I(\pi(t)) \right) = \inf_{\pi \in \Pi'} \max_t I(\pi(t)) - \sup_{\pi \in \Pi'} \min_t I(\pi(t))$$

for bounded, continuous $I$!

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**Properties**

- **Geodesic distance**
  - Gradient based (noise, blur sensitive)
  - A parameter gives spatial and intensity distance trade-off

- **Fuzzy distance**
  - Favors high/low intensity values
  - $\Rightarrow$ suited for gradient magnitude image (noise sensitive)
  - Developed for fuzzy membership values

- **Fuzzy connectedness**
  - Gradient based (noise, blur sensitive)
  - No spatial distance term
  - Support for apriori information

- **Minimum barrier distance**
  - Noise and blur stability
  - No spatial distance term
  - Interval weighted graph
Algorithms

• Efficient algorithms needed in order to
  -process large amount of information and
  -minimize execution time

Repeated raster scan until convergence.
Drawback: Highly image content dependent
Slow convergence in some cases
Algorithms
Repeated raster scan until convergence.
Scan 1, forward pass
Algorithms

Repeated raster scan until convergence.

Scan 1, forward pass
Algorithms
Repeated raster scan until convergence.
After scan 1, forward pass

Algorithms
Repeated raster scan until convergence.
After scan 1, backward pass
Algorithms
Repeated raster scan until convergence.
After scan 2, forward pass
Algorithms

Dijkstra’s algorithm / wavefront propagation

A path cost function is *monotonic incremental* if

- \( f(\pi \cdot (s, t)) = f(\pi) \circ (s, t) \), where \( \circ \) satisfies
  - \( x' \geq x \Rightarrow x' \circ (s, t) \geq x \circ (s, t) \)
  - \( x \circ (s, t) \geq x \)

In other words:

- Adding the same element to two paths preserves the order of the cost of the paths.
- The path cost function is non-decreasing (adding an element to a path gives higher or equal cost).

References:

Algorithms
Dijkstra’s algorithm / wavefront propagation

Monotonic incremental:
• Curve length
• Curve integral (if non-negative)
• Max/min

Not monotonic incremental:
• Minimum barrier distance

Minimum barrier

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Image, intensity values

Graph representation
Minimum barrier

Graph representation

Interval weighted graph

Minimum barrier
Dijkstra approximation

Interval weighted Dijkstra
Minimum barrier
Dijkstra approximation

Interval weighted Dijkstra

Result from interval
weighted Dijkstra

The cost of this path is 3!
Minimum barrier Dijkstra approximation

Another approximation: \( \min \max - \max \min \)

i) minimize max

Graph representation  Weighted graph  Result
Another approximation: \( \min \max - \max \min \)

ii) maximize min

Graph representation | Weighted graph | Result
--- | --- | ---

Another approximation: \( \min \max - \max \min \)

Min max | Max min | Difference
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Another approximation: \( \min \max - \max \min \)

\[
\begin{array}{ccc}
0 & 2 & 2 \\
1 & 0 & 4 \\
4 & 3 & 0 \\
\end{array}
\quad
\begin{array}{ccc}
0 & 2 & 4 \\
1 & 1 & 4 \\
4 & 3 & 3 \\
\end{array}
\]

Approximation
Correct

Minimum path cost/Distance transform

Seed point
Minimum path cost transform

1) Geodesic distance, \( f(m, n) = \sqrt{(m - n)^2 + 1} \)

2) Curve integral/Fuzzy distance, \( f(m, n) = \frac{|m+n|}{2} \)
Minimum path cost transform

3) Gradient magnitude ('inverted fuzzy connectedness')
\[ f(m, n) = |m - n| \]

Minimum path cost transform

4) Minimum barrier, \( f(m, n) = (\min(m, n), \max(m, n)) \)
Interactive (semi-automatic) image processing

Human
Complex patterns
Interpret based on experience

Computer
Quantitative analysis
Objective
dFast

Segmentation

MR image
Interactive segmentation: The user adds background and object seed points. Labels are propagated together with cost values.
Segmentation

Interactive segmentation: The user adds seed points and the segmentation is updated locally.

Segmentation of volume images

MR image

MR, volume image
Segmentation of volume images

Seed points  Segmentation result

Summary

Minimum path cost methods
• Geodesic distance
• Fuzzy connectedness.
• Fuzzy distance
• Minimum barrier distance

Efficient computation by wave-front propagation
Some applications in segmentation