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Binary Codes for Counting Digital Topologies in $\mathbf{Z}^n$

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We address an open problem for the computation of exact numbers of digital topologies (as defined for image analysis, see [2]) in n-dimensional orthogonal grid space $\mathbf{Z}^n$, for $n \geq 2$. These topologies are defined by Hamilton loops on the n-dimensional hypercube

$$\mathbf{B}_n = \{\boldsymbol{p}_i = (\epsilon_1^i, \epsilon_2^i, \cdots, \epsilon_n^i)^\top : \ 0 \leq i \leq 2^n - 1 \ \wedge \ \epsilon_k^i \in \{0,1\}\} . \tag{1}$$

$\mathbf{B}_n$ can be represented by a graph with 2^n vertices labeled from 0 to $2^n - 1$ in such a way that there is an edge between any two nodes iff the binary representation of their labels differs in exactly one bit. A k-dimensional hypercube consists of two $(k-1)$-dimensional hypercubes with edges between corresponding ('identical') vertices in both $(k-1)$-dimensional hypercubes. The hypercube graph is complete with valency n. The n neighbors of vertex $a_{n-1}a_{n-2}\cdots a_0$ are $a_{n-1}a_{n-2}\cdots \bar{a}_i \cdots a_0$.

A Hamilton loop is defined by orienting all edges between vertices $\boldsymbol{p}_i$ and $\boldsymbol{p}_j$ of the hypercube, representing a closed path $\boldsymbol{p}_0 \cdots \boldsymbol{p}_{2^n-1}, \boldsymbol{p}_0$. Let a_{ij} be an encoding of these orientations with $a_{ij} = 1$ if there is an oriented edge from $\boldsymbol{p}_i$ to $\boldsymbol{p}_j$, $a_{ij} = -1$ if there is an oriented edge from $\boldsymbol{p}_j$ to $\boldsymbol{p}_i$, and $a_{ij} = 0$ otherwise. It follows that

$$\sum_{i=0}^{2^n-1} a_{i,i+1} = 0 , \quad \text{with } a_{2^n-1,2^n} = a_{2^n-1,0} \tag{2}$$

and $a_{i,i+1}a_{i+1,i+2}a_{i+2,i+3} \neq 111$, for any vertex i of the hypercube.

The enumeration problem of topologies in $\mathbf{Z}^n$ has been formulated in combinatorics in two different ways [1]:

(1) determine all Hamilton loops on the n-dimensional hypercube, or

(2) determine any a_{ij}-sequence such that the sum of codes is zero excluding triples $a_{i,i+1}a_{i+1,i+2}a_{i+2,i+3} = 111$.

Approach **(1)** has been solved for $n = 2, 3, 4$, setting

$$p_i = \epsilon_1^i + 2\epsilon_2^i + \cdots + 2^{n-1}\epsilon_n^i . \tag{3}$$

For $n = 2$ the only possible Hamilton (or Euler) loop is 0132, a Hamilton loop for $n = 3$ is 01326754, and for $n = 4$ we have (in hexagonal numbers)

$01326754CDFEAB98$ as a possible Hamilton loop. The second approach **(2)** may utilize the definition

$$H_{k+1} = H_2 \otimes H_k \text{ , with } H_2 = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \qquad (4)$$

of Hadamard matrices which do not have row vectors with three 1's in succession.

The k-bit binary reflected Gray code, denoted by G_k, is recursively defined by $G_1 = \{0, 1\}$, $G_i = \{g_0, g_1, \cdots g_{2^i-1}\}$, and

$$G_{i+1} = \{0g_0, 0g_1, \cdots 0g_{2^i-1}, 1g_{n-1}, 1g_{2^i-2}, \cdots, 1g_0\} \ .$$

The binary reflected Gray code is periodic. It defines a bijection from elements of a Hadamard matrix H_n to nodes on a hypercube graph $\mathbf{B}_n$. Hadamard matrices contain redundancies for the expression of topologies in $\mathbf{Z}$.

We illustrate representations of hypercubes by Hadamard matrices, using symbols $+$ and $-$ to stand for 1 and -1. The matrices for $n = 1, 2, 3, 4$ are as follows:

$$H_1 = \begin{matrix} + & + \\ + & - \end{matrix} \qquad\qquad H_2 = \begin{matrix} + & + & + & + \\ + & - & + & - \\ + & + & - & - \\ + & - & - & + \end{matrix} \qquad (5)$$

$$H_3 = \begin{matrix} + & + & + & + & + & + & + & + \\ + & - & + & - & + & - & + & - \\ + & + & - & - & + & + & - & - \\ + & - & - & + & + & - & - & + \\ + & + & + & + & - & - & - & - \\ + & - & + & - & - & + & - & + \\ + & + & - & - & - & - & + & + \\ + & - & - & + & - & + & + & - \end{matrix} \qquad (6)$$

$$
H_4 = \begin{array}{cccccccccccccccc}
+ & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + \\
+ & - & + & - & + & - & + & - & + & - & + & - & + & - & + & - \\
+ & + & - & - & + & + & - & - & + & + & - & - & + & + & - & - \\
+ & - & - & + & + & - & - & + & + & - & - & + & + & - & - & + \\
+ & + & + & + & - & - & - & - & + & + & + & + & - & - & - & - \\
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+ & + & - & - & - & - & + & + & + & + & - & - & - & - & + & + \\
+ & - & - & + & - & + & + & - & + & - & - & + & - & + & + & - \\
+ & + & + & + & + & + & + & + & - & - & - & - & - & - & - & - \\
+ & - & + & - & + & - & + & - & - & + & - & + & - & + & - & + \\
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+ & - & - & + & + & - & - & + & - & + & + & - & - & + & + & - \\
+ & + & + & + & - & - & - & - & - & - & - & - & + & + & + & + \\
+ & - & + & - & - & + & - & + & - & + & - & + & + & - & + & - \\
+ & + & - & - & - & - & + & + & - & - & + & + & + & + & - & - \\
+ & - & - & + & - & + & + & - & - & + & + & - & + & - & - & +
\end{array} \qquad (7)
$$

If we eliminate all rows containing triplets $+++$, we obtain an uperbound $2^n - 2^{n-2}$ for the number of possible digital topologies in $\mathbf{Z}^n$.

The open problem is: Improve this upper bound or show that this is the exact number of digital topologies in $\mathbf{Z}^n$.

References

[1] R. L. Graham, M. Gröschel, and L Lovaász, *Handbook of Combinatrics,* Vols. 1 and 2, Horth-Holland, Elsevier;Amsterdam, 1995.

[2] K. Voss, *Discrete Images, Objects, and Functions,* Springer-Verlag;Berlin, 1993.