Vertices of the digital line/(hyper)plane segment polytope

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Abstract

We consider the following problem. Given an 8-digital straight line segment (8-DSS) of length n, estimate the number of vertices of its convex hull. Taking advantage of known results from polyhedral combinatorics, we answer this question and its generalization to higher dimensions. We also address algorithmic issues and provide relevant references.

Keywords: *digital line segment, digital (hyper)plane segment, convex hull, digital line segment polytope, digital (hyper)plane segment polytope, knapsack polytope*

1 Introduction

Digital geometry has its theoretical roots in a number of classical subjects, such as number theory, geometry of numbers, graph theory, cell complexes, and others. An important direction of research is seen in adapting notions, theoretical constructs, open questions, and results from previous studies to digital geometry framework. This way digital geometry may achieve more solid theoretical foundations and integration with other disciplines.

The present note provides an example for transfer of results from polyhedral combinatorics to digital geometry.

We consider the following problem. Given an 8-digital straight line segment (8-DSS) of length n (i.e., involving n pixels), estimate the number of vertices of its convex hull. An upper bound $O(\log n)$ is known in digital geometry community. Occasionally, it has been referred to a work of J. Koplowitz on adjacent pairs characterization of digital lines, although there formulation or proof of such a result is not available. Finding a lower bound is considered as an open question. (See [7], Question 14 in the list of open questions proposed at the Dagstuhl seminar 2004). In what follows, we provide an answer to the above question and to its generalization to higher dimensions. Basically, the considered question is equivalent to the one of evaluating the number of knapsack polytope vertices, the latter being a well-studied problem in integer programming and polyhedral combinatorics. We also address algorithmic issues and provide relevant references for future citations.

2 Preliminaries

2.1 Digital lines and (hyper)planes

A *digital line* can be defined as a set $L(a_1, a_2, b) = \{(x_1, x_2) \in \mathbb{Z}^2 | 0 < (\leq)\ a_1x_1 + a_2x_2 + b + \lfloor \max(|a_1|, |a_2|)/2 \rfloor \leq (<) \max(|a_1|, |a_2|)\}$, where $a_1, a_2, b \in \mathbb{Z}$ (see [8]). $L(a_1, a_2, b, \max(|a_1|, |a_2|))$ can be regarded as a discretization of a straight line with equation $ax_1 + ax_2 + b + \lfloor \max(|a_1|, |a_2|)/2 \rfloor = 0$. The above definition extends to higher dimensions: A *digital hyperplane* is a set of the form

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$H(a_1, \ldots, a_n, b) = \left\{(x_1, \ldots, x_n) \in \mathbb{Z}^n | 0 < (\leq)\ a_1x_1 + \cdots + a_nx_n + b + \left\lfloor \frac{|a|_{\max}}{2} \right\rfloor \leq (<)\ |a|_{\max}\right\}$, where $|a|_{\max} = \max(|a_1|, \ldots, |a_n|)$. $H(a_1, \ldots, a_n, b, |a|_{\max})$ can be regarded as a discretization of a hyperplane $a_1x_1 + \cdots + a_nx_n + b + \left\lfloor \frac{|a|_{\max}}{2} \right\rfloor = 0$.

2.2 Knapsack polytope

Let $a = (a_1, \ldots, a_n) \in \mathbb{Z}_+^n$ and $b \in \mathbb{Z}_+$. Denote by $M(a, b)$ the set of integer nonnegative solutions to the inequality $\sum_{i=1}^n a_ix_i \leq (<)\ b$, and by $K(a, b)$ the convex hull of $M(a, b)$. $K(a, b)$ is called the *knapsack polytope*. Let $V_K(a, b)$ be the set of its vertices and $|V_K(a, b)|$ its cardinality. Analogously we can consider an "upper" knapsack polytope defined by constraint of the form $\sum_{i=1}^n a_ix_i > (\geq)\ b$, $x_1, \ldots, x_n \geq 0$, for $a_1, \ldots, a_n, b > 0$.

3 Digital line/plane segment polytope

Now let L be a digital line segment. According to Section 2.1, L is a set of integer points $\{p^1, p^2, \ldots, p^m\}$, $p^i = (p_1^i, p_2^i)$, satisfying for some positive integers a_1, a_2, b the condition $0 < a_1p_1^i + a_2p_2^i + b + \lfloor \max(|a_1|, |a_2|)/2 \rfloor \leq \max(|a_1|, |a_2|)$. W.l.o.g., we may assume that L is a digitization of a line segment with end-points on the positive axes x_1 and x_2. Let $a_1 \leq a_2$. Then we clearly have $m = \lfloor \frac{b}{a_1} \rfloor$, $p^1 = (\lfloor \frac{b}{a_1} \rfloor, 0)$, and $p^m = (0, \lfloor \frac{b}{a_2} \rfloor)$.

We call the convex hull of $\{p^1, p^2, \ldots, p^m\}$ a *digital line segment polytope* and denote it $D(a, b)$. We denote the set of its vertices by $V_D(a, b)$ and their number by $|V_D(a, b)|$.

We make the natural assumption that a (hyper)plane segment in $\mathbb{R}^n$ is a simplex in $\mathbb{R}^n$. Then a digital (hyper)plane segment polytope is defined analogously to a digital line segment polytope.

Since a digital line/(hyper)plane polytope $D(a, b)$ is defined by two "parallel" knapsack constraints, it is easy to see that upper and lower bounds on the number of vertices of a knapsack polytope $K(a, b)$ will also hold for the set of vertices of $D(a, b)$.

4 Upper and lower bounds for $|V_K(a, b)|$ and $|V_D(a, b)|$

Various upper bounds on $|V_K(a, b)|$ are available in the literature (see, e.g., a recent survey by Zolotych [11]). As a matter of fact, they are all of the same order and differ only by form. In what follows, we provide a short review of some well-known bounds on $|V_K(a, b)|$, which imply corresponding bounds on $|V_D(a, b)|$.

The problem of evaluating $|V_K(a, b)|$ is a rare case when a tight lower bound appeared earlier that upper bounds. In 1970 Rubin [9] gave an example showing that a knapsack polytope may have arbitrary many vertices. Although not explicitly indicated, that example provides an $\Omega(\log \frac{b}{a_1}) = \Omega(\log m)$ lower bound. It is given by a linear constraint of the form $F_{2k}x_1 + F_{2k+1}x_2 \leq F_{2k+1}^2 - 1$, where F_{2k} and F_{2k+1} are two consecutive Fibonacci numbers. A knapsack polytope determined by the above inequality has $k+3$ vertices. Since $k = \Omega\left(\log \left\lfloor \frac{F_{2k+1}^2 - 1}{F_{2k}} \right\rfloor\right)$, the stated lower bound holds. (In terms of digital lines, a digital line segment $L(F_{2k}, F_{2k+1}, F_{2k+1}^2 - 1)$, $x_1, x_2 \geq 0$, has $m = \left\lfloor \frac{F_{2k+1}^2 - 1}{F_{2k}} \right\rfloor$ pixels and $k + 3 = \Theta(\log m)$ vertices).

In early 80's a number of upper bound for arbitrary dimension n were proposed (see, e.g., [5, 10]). They are of the form $O((\log \frac{b}{a_1})^n)$ and are not tight for fixed dimensions. For instance, for $n = 2$, we get $|V_K(a, b)| = O((\log \frac{b}{a_1})^2)$.

The lower bound $\Omega(\log \frac{b}{a_1})$ is tight since it matches an upper bound $O(\log \frac{b}{a_1})$, i.e., for $n = 2$, we have $|V_K(a,b)| = \Theta(\log \frac{b}{a_1})$.

To our knowledge of the literature, the first rigorous proof of the upper bound $|V_K(a,b)| = O(\log \frac{b}{a_1})$ was given by Brimkov in 1984 in his MS thesis [2]. Moreover, an $O(C(n)(\log \frac{b}{a_1})^{n-1})$ upper bound for arbitrary dimension n was proved there. Here $C(n)$ is a function depending only on n. (This, in particular, implies the tight upper bound for $n = 2$.) In 1987 these results appeared in [3]. Since the above works were published in Bulgarian language, they remained unknown.

In 1992 Morgan [6] published the same result. In the related literature this last publication is cited as the first one where the power of the bounding polynomial was reduced to $n - 1$, although it appeared several years later than [2, 3].

In 1992 Bárány, Hove and Lovasz [1] closed the issue by proving a lower bound in arbitrary dimension. They showed that $|V_K(a,b)| = \Omega(C'(n)\phi^{n-1})$, where ϕ is the input bit-size and $C'(n)$ a function depending only on n. This lower bound together with the above-mentioned upper bound $O(C(n)(\log \frac{b}{a_1})^{n-1})$ imply a tight bound $|V_K(a,b)| = \Theta(\phi^{n-1})$, within a constant factor depending only on n.

Note that the above-listed bounds hold also for the integer points in an arbitrary polytope determined by a system of linear inequalities $Ax \leq b$.

One can easily interpret the above results in terms of digital hyperplanes.

As a final remark we mention that the approach of Hayes and Larman from [5] is constructive and implies an algorithm for generation of the knapsack polytope vertices in arbitrary dimension. The algorithm is quasipolynomial, i.e., it is polynomial when the dimension n is fixed. In [4] Brimkov modified Hayes-Larman's algorithm in a way that the degree of the polynomial measuring the time complexity was decreased by one.

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