Arithmetic discrete planes are quasicrystals

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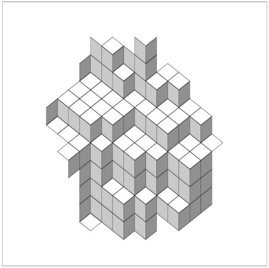
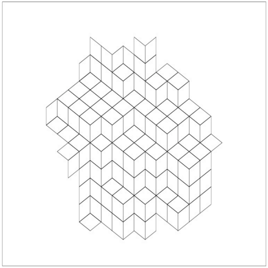
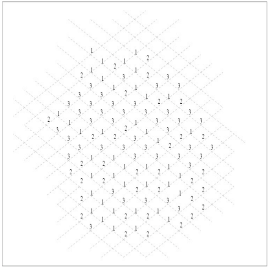
From discrete geometry to word combinatorics...







...via tilings and quasicrystals

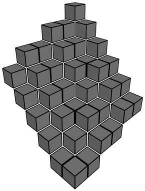
Arithmetic discrete planes [Reveillès'91]

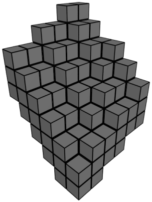
Let $\vec{v} \in \mathbb{R}^d$, $\mu, \omega \in \mathbb{R}$.

The arithmetic discrete plane $\mathfrak{P}(\vec{v}, \mu, \omega)$ is defined as

$$\mathfrak{P}(\vec{v}, \mu, \omega) = \{\vec{x} \in \mathbb{Z}^d \mid 0 \leq \langle \vec{x}, \vec{v} \rangle + \mu < \omega\}.$$

- μ is the translation parameter.
- ω is the width.
- If $\omega = \max_i\{|v_i|\} = ||\vec{v}||_\infty$, then $\mathfrak{P}(\vec{v}, \mu, \omega)$ is said naive.
- If $\omega = \sum_i |v_i| = ||\vec{v}||_1$, then $\mathfrak{P}(\vec{v}, \mu, \omega)$ is said standard.





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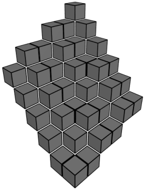
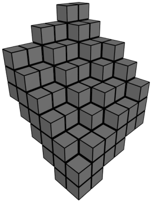




We work here with $d = 2, 3$. The vector $\vec{v}$ is assumed to be a nonzero vector with nonnegative coordinates. We consider here integer as well as irrational parameters $\vec{v}, \mu, \omega$.

What are quasicrystals?

Quasicrystals are atomic structures discovered in 84 [Shechtman-Blech-Gratias-Cahn] that are both ordered and nonperiodic.

- Like crystals, quasicrystals produce Bragg diffraction.
- Diffraction comes from regular spacing and long-range order.

A large family of models of quasicrystals is produced by cut and project schemes:

projection of a plane slicing through a higher dimensional lattice

- The order comes from the lattice structure.
- The nonperiodicity comes from the normal vector of the plane.

Cut and project scheme

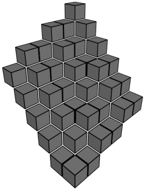
Cut and project scheme:

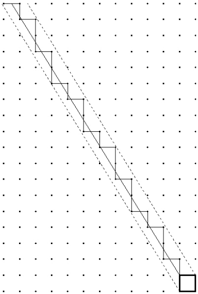
projection of a plane slicing through the lattice $\mathbb{Z}^d$

- Cutting step

$$\mathfrak{P}(\vec{v}, \mu, \omega) = \{\vec{x} \in \mathbb{Z}^3 \mid 0 \leq \langle \vec{x}, \vec{v} \rangle + \mu < \omega\}.$$

The selection window is $[0, \omega]$.



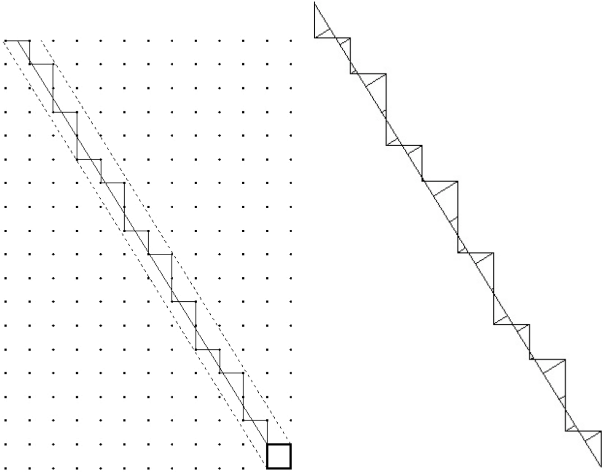


Cut and project scheme

Cut and project scheme:

projection of a plane slicing through the lattice $\mathbb{Z}^d$

- Projection step



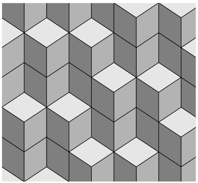
Cut and project scheme

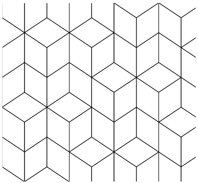
Cut and project scheme:

projection of a plane slicing through the lattice $\mathbb{Z}^d$

- Projection step

Let π_0 be the orthogonal projection onto P_0: $\langle \vec{x}, (1,1,1) \rangle = 0$.
We project by π_0 the arithmetic discrete plane $\mathfrak{P}(\vec{v}, \mu, \omega)$.





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One gets a set of points of P_0 which is a Delone set, *i.e.*, a set that is both

- relatively dense: there exists $R > 0$ such that any Euclidean ball of P_0 of radius R contains a point of this set,
- uniformly discrete: there exists $r > 0$ such that any ball of radius r contains at most one point of this set.

About the selection window

- Configurations correspond to subintervals of the selection window.

- By playing with the selection window, we can go from discrete planes to discrete planes

$\rightsquigarrow$ substitution rules

About the selection window

Configurations correspond to subintervals of the selection window...

....by looking at discrete planes as

- tilings
- multidimensional words

Back to the projection step: a tiling by lozenges....

- We work in the standard case $\omega = ||\vec{v}_1||$.

- We associate with the quasicrystal $\pi_0(\mathfrak{P}(\vec{v}, \mu, \omega))$ a tiling $T(\vec{v}, \mu, \omega))$ of the plane by three kinds of lozenges, obtained by connecting points of the quasicrystal with edges.

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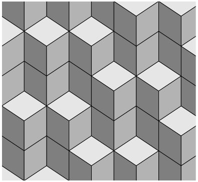

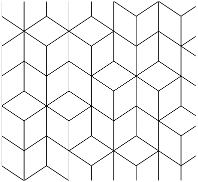

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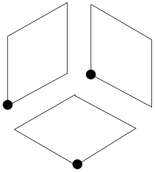
- A tiling by translation of the plane by a set T of (proto)tiles is a union of translates of elements of T that covers the full space, with any two tiles intersecting either on an empty set, or on a vertex or on an edge.



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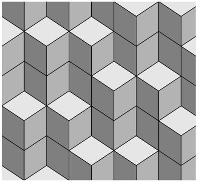
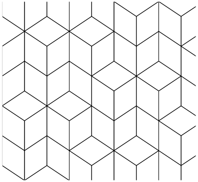
- We associate with the arithmetic discrete plane $\mathfrak{P}(\vec{v}, \mu, \omega)$ a surface $\mathcal{P}(\vec{v}, \mu, \omega)$ in $\mathbb{R}^3$ called stepped plane defined as the union of translates of faces of the unit cube whose vertices belong to $\mathfrak{P}(\vec{v}, \mu, \omega)$.





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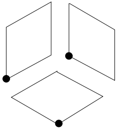


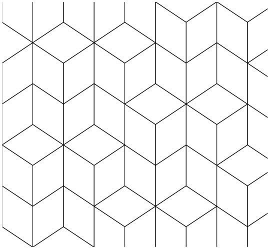
We now pick for each tile T_i a particular vertex, called its distinguished vertex.
One has a one-to-one correspondence between tiles T_i of the tiling and faces F_i of the stepped plane

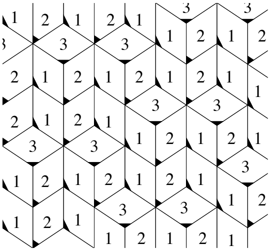
Fact: The set of distinguished vertices of $\mathfrak{P}(\vec{v}, \mu, \omega)$ is a lattice.

..and a multidimensional word

Fact: The set of distinguished vertices of $\mathfrak{P}(\vec{v}, \mu, \omega)$ is a lattice.

Since the set of distinguished vertices is a lattice that can be assimilated to $\mathbb{Z}^2$, we can code as a $\mathbb{Z}^2$-word over the alphabet $\{1, 2, 3\}$ any arithmetic discrete plane.





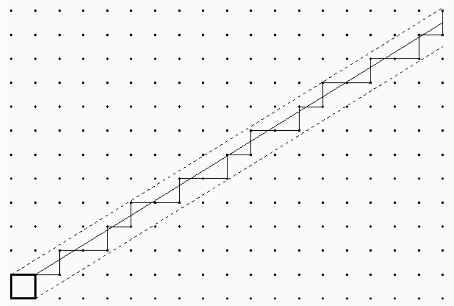
lattice structure $\rightsquigarrow$ some regularity

Two-dimensional word combinatorics

An arithmetic discrete plane can be coded as

1	2	1	2	1	2	3	1	2	1	2	3	1	3
3	1	3	1	2	1	2	3	1	2	1	2	1	2
2	1	2	3	1	2	1	2	3	1	3	1	2	1
1	2	1	2	3	1	3	1	2	1	2	3	1	2
3	1	2	1	2	1	2	3	1	2	1	2	3	1
2	3	1	3	1	2	1	2	3	1	2	1	2	1
1	2	1	2	3	1	2	1	2	3	1	3	1	2
3	1	2	1	2	3	1	3	1	2	1	2	3	1

Discrete lines and Sturmian words



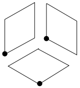
One can code such a discrete line (Freeman code) over the two-letter alphabet $\{0, 1\}$.
One gets a Stumian word $(u_n)_{n \in \mathbb{N}} \in \{0, 1\}^{\mathbb{N}}$

0100101001001010010100100101

Selection window and faces

We want now to localize with respect to the value $\langle \vec{x}, \vec{v} \rangle$ in the selection window $[0, \omega)$ the distinguished vertices of faces of a given type

$$0 \leq \langle \vec{x}, \vec{v} \rangle + \mu < ||\vec{v}||_1 = v_1 + v_2 + v_3.$$



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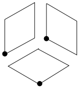

- Faces of type 1 Assume first that

$$0 \leq \langle \vec{x}, \vec{v} \rangle + \mu < v_1.$$

Then

$$\vec{x} + \vec{e}_2,\ \vec{x} + \vec{e}_3,\ \vec{x} + \vec{e}_2 + \vec{e}_3$$

all belong to $\mathfrak{P}(\vec{v}, \mu, \omega)$. Hence the full face $F_1 + \vec{x}$ is included in $\mathcal{P}(\vec{v}, \mu, \omega)$.

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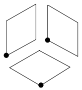


- Faces of type 2 Assume

$$v_1 \leq \langle \vec{x}, \vec{v} \rangle + \mu < v_1 + v_2.$$

Then

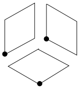
$$\vec{x} - \vec{e}_1,\ \vec{x} + \vec{e}_3,\ \vec{x} - \vec{e}_1 + \vec{e}_3$$

all belong to $\mathfrak{P}(\vec{v}, \mu, \omega)$. Hence the full face $F_2 + \vec{x}$ is included in $\mathcal{P}(\vec{v}, \mu, \omega)$.

Selection window and faces

We want now to localize with respect to the value $\langle \vec{x}, \vec{v} \rangle$ in the selection window $[0, \omega)$ the distinguished vertices of faces of a given type

$$0 \leq \langle \vec{x}, \vec{v} \rangle + \mu < ||\vec{v}||_1 = v_1 + v_2 + v_3.$$



- Faces of type 3 Assume

$$v_1 + v_2 \leq \langle \vec{x}, \vec{v} \rangle + \mu < v_1 + v_2 + v_3.$$

Then

$$\vec{x} - \vec{e}_1, \ \vec{x} - \vec{e}_2, \ \vec{x} - \vec{e}_1 - \vec{e}_2$$

all belong to $\mathfrak{P}(\vec{v}, \mu, \omega)$. The face $F_3 + \vec{x}$ is included in $\mathcal{P}(\vec{v}, \mu, \omega)$.

Selection window and faces

We want now to localize with respect to the value $\langle \vec{x}, \vec{v} \rangle$ in the selection window $[0, \omega)$ the distinguished vertices of faces of a given type

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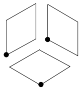

We have cut the selection interval $[0, ||\vec{v}||_1)$ into three subintervals,

$$I_1 = [0, v_1), \;\; I_2 = [v_1, v_1 + v_2), \;\; I_3 = [v_1 + v_2, v_1 + v_2 + v_3),$$

each of them corresponding to the occurrences of the distinguished vertex of a particular type of face.

$\rightsquigarrow$ configurations

Rational vs irrational arithmetic discrete planes

The arithmetic discrete plane $\mathfrak{P}(\vec{v}, \mu, \omega)$ is defined as

$$\mathfrak{P}(\vec{v}, \mu, \omega) = \{(x, y, z) \in \mathbb{Z}^3 \mid 0 \leq v_1 x + v_2 y + v_3 z + \mu < \omega\}.$$

Remark

- Totally irrational planes: $\dim_{\mathbb{Q}}(v_1, v_2, v_3) = 3$.
- Irrational planes (the intermediate case): $\dim_{\mathbb{Q}}(v_1, v_2, v_3) = 2$.
- Rational planes: $\dim_{\mathbb{Q}}(v_1, v_2, v_3) = 1$. One can choose $v_1, v_2, v_3, \mu, \omega \in \mathbb{Z}$ with

$$\gcd(v_1, v_2, v_3) = 1 \text{ (Bezout's Lemma).}$$

The determination of the frequencies of factors is deduced from the properties of equidistribution for the sequence

$$((m v_1 + n v_2) \bmod \omega)_{(m,n)) \in \mathbb{Z}^2}.$$

Configurations

- A configuration of the tiling $T(\vec{v}, \mu, \omega)$ is an edge-connected finite union of lozenge tiles contained in the tiling.
- Liftings in $\mathfrak{P}(\vec{v}, \mu, \omega)$ of configurations correspond to usual local configurations of discrete planes.
- We associate with the configuration C the set I_C of the selection window defined as the closure of the set

$$\{\langle \vec{x}, \vec{v} \rangle + \mu \mid \vec{y} = \pi_0(\vec{x}),\ \vec{x} \in \mathfrak{P}(\vec{v}, \mu, \omega),\ C \text{ occurs in } T(\vec{v}, \mu, \omega) \text{ at } \vec{y}\}.$$

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  - The set I_C is an interval if the dimension of the $\mathbb{Q}$-vector space generated by the coordinates of $\vec{v}$ is at least 2. We use the denseness in the acceptance window $[0,\omega)$ of $(\langle\vec{x},\vec{v}\rangle)_{\vec{x}\in\mathbb{Z}^2}$.
  - If $\vec{v}$ has integer coprime entries and μ is also an integer, I_C is a set of consecutive integers. We use in this latter case Bezout's lemma.

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Example Consider $C = T_1 \cup (T_1 + \vec{e}_3)$. Recall that

$$I_1 = [0, v_1),\ I_2 = [v_1, v_1 + v_2),\ I_3 = [v_1 + v_2, v_1 + v_2 + v_3).$$

Configuration C occurs at $\vec{y} = \pi_0(\vec{x})$ if and only if

$$\langle \vec{x}, \vec{v} \rangle + \mu \in I_1 \text{ and } \langle \vec{x} + \vec{e}_3, \vec{v} \rangle + \mu = \langle \vec{x}, \vec{v} \rangle + v_3 + \mu \in I_1,$$

$$\langle \vec{x}, \vec{v} \rangle + \mu \in I_1 \cap (I_1 - v_3).$$

Hence

- $I_C \neq \emptyset$ if and only if $v_1 > v_3$.
- If $v_1 > v_3$, then $I_C = [0, v_1 - v_3)$

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We thus have here again divided the selection window $[0, \omega)$ into intervals I_C associated with configurations C.

Applications

Two discrete planes with the same normal vector and same width have the same configurations.

We also deduce information on the

- number of configurations/factors of a given size (enumeration)
- frequencies (probabilities)

See for instance [B.-Vuillon] and more generally [Daurat-Tajine-Zouaoui DGCI'09]

Application: repetitivity

• The radius of a configuration is defined as the minimal radius of a disk containing this configuration.

• Two configurations are said identical if they only differ by a translation vector.

• A tiling is said repetitive if for every configuration C there exists a positive number R such that every configuration of radius R contains C.

Configurations appear “with bounded gaps”. Repetitive tilings can be considered as ordered structures.

Configurations = words, repetitivity = uniform recurrence

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Configurations = words, repetitivity = uniform recurrence

Proof of the repetitivity: Let C be a given configuration that occurs in the tiling T. We consider the interval associated with C.
Given any interval I of $\mathbb{R}/\mathbb{Z}$, the sequence $(n\alpha)_{n\in\mathbb{N}}$ mod 1 enters the interval I with bounded gaps, that is, there exists $N \in \mathbb{N}$ such that any sequence of N successive values of the sequence contains a value in I.

From discrete planes to tilings via... number theory

Fact: Arithmetic discrete planes are repetitive.

Repetitivity function: Let N be the smallest integer N such that every ball of radius N in the tiling contains all configurations of radius n. We set $R(n) := N$.

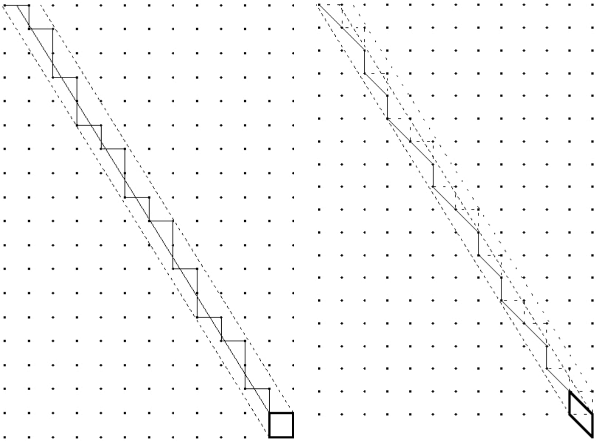
Linear repetitivity: there exists C such that $R(n) \leq Cn$ for all n.

Open problem: Characterize the discrete planes which have linear repetitivity.

Discrete lines: one has linear repetitivity iff and the slope of the line has bounded partial quotients in its continued fraction expansion.

Repetitivity is a measure of disorder.

Playing with the selection window



As an example, see [Domenjoud-Jamet-Toutant DGCI'09]

By playing with the selection window....

....we would like to be able to

- generate discrete planes
- recognize discrete planes: given a set of points in $\mathbb{Z}^3$, is it contained in an arithmetic discrete plane?

$\rightsquigarrow$ Hierarchical structure/substitution rules

Toward multidimensional continued fractions

- We have been so far able to describe properties of arithmetic discrete planes sharing the same normal vector $\vec{v}$ by cutting the selection window into intervals associated with configurations.
- We now want to be able to relate two discrete planes with different normal vectors $\vec{v}$ and $\vec{v}'$.
- We focus on the case $\vec{v} = M\vec{v}'$, where M is a 3 by 3 square matrix with entries in $\mathbb{N}$ having determinant equal to 1 or -1.

Continued fractions

One represents any positive real number α as

$$\alpha = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cdots}}}$$

in order to find good rational approximations of α.

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In matricial terms this gives

$$\begin{pmatrix} p_{n+1} & p_n \\ q_{n+1} & q_n \end{pmatrix} = \begin{pmatrix} a_1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a_2 & 1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} a_{n+1} & 1 \\ 1 & 0 \end{pmatrix}$$

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$$\begin{pmatrix} 1 \\ \alpha \end{pmatrix} \sim \begin{pmatrix} a_1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a_2 & 1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} a_n & 1 \\ 1 & 0 \end{pmatrix} \cdots$$

One approximates a direction $(1, \alpha)$ by a succession of nested cones.

Multidimensional continued fractions

If we start with two parameters (α, β), one looks for two rational sequences (p_n/q_n) et (r_n/q_n) with the same denominator that satisfy

$$\lim p_n/q_n = \alpha, \lim r_n/q_n = \beta.$$

There is no canonical multidimensional continued fraction.

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There is no canonical multidimensional continued fraction.

Which kind of continued fraction algorithm can we use in discrete geomery to describe

- discrete lines in $\mathbb{R}^3$
- discrete planes in $\mathbb{R}^3$?

Unimodular multidimensional continued fractions

Let $X \subset \mathbb{R}^d$. A d-dimensional continued fraction map over X is a map $T : X \to X$ such that $T(X) \subset X$ and, for any $\vec{x} \in X$, there is a matrix $M(\vec{x})$ in $GL(d, \mathbb{Z})$ satisfying:

$$\vec{x} = M(\vec{x}).T(\vec{x}).$$

The associated continued fraction algorithm consists in iteratively applying the map T on a vector $\vec{x} \in X$. This yields the following sequence of matrices, called the continued fraction expansion of $\vec{x}$:

$$(M(T^n(\vec{x})))_{n \in \mathbb{N}}.$$

If the matrices have nonnegative entries, the algorithm is said nonnegative.

Jacobi-Perron algorithm

- Its projective version is defined on the unit square $[0,1) \times [0,1)$ by:

$$(\alpha, \beta) \mapsto \left(\frac{\beta}{\alpha} - \left\lfloor \frac{\beta}{\alpha} \right\rfloor, \frac{1}{\alpha} - \left\lfloor \frac{1}{\alpha} \right\rfloor \right) = (\{\beta/\alpha\}, \{1/\alpha\}) .$$

- Its linear version is defined on the positive cone $X = \{(a, b, c) \in \mathbb{R}^3 | 0 \leq a, b < c\}$ by:

$$T(a, b, c) = (b - \lfloor b/a \rfloor a, c - \lfloor c/a \rfloor a, a).$$

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- We set $(a_0, b_0, c_0) := (a,b,c)$ and $(a_{n+1}, b_{n+1}, c_{n+1}) := T^n(a_n, b_n, c_n)$, for $n \in \mathbb{N}$. Let $B_{n+1} = \lfloor b_n/a_n \rfloor a_n$, $C_n = \lfloor c_n/a_n \rfloor$. One has

$$\begin{pmatrix} a_n \\ b_n \\ c_n \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & B_{n+1} \\ 0 & 1 & C_{n+1} \end{pmatrix} \begin{pmatrix} a_{n+1} \\ b_{n+1} \\ c_{n+1} \end{pmatrix}.$$

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$$\begin{pmatrix} a_n \\ b_n \\ c_n \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & B_{n+1} \\ 0 & 1 & C_{n+1} \end{pmatrix} \begin{pmatrix} a_{n+1} \\ b_{n+1} \\ c_{n+1} \end{pmatrix}.$$

- The idea is now to consider the expansion of a given normal vector $\vec{v} \in X$. Let $\vec{v}^{(n)}$ stand for $T^n(\vec{v})$. One expands $\vec{v}$ as

$$\vec{v} = M_{B_1,C_1} \cdots M_{B_n,C_n} \vec{v}^{(n)}.$$

Generalized substitutions

- We assume that we are in the standard case $\omega = ||\vec{v}||_1$.
- We would like to give a description of $\mathfrak{P}(\vec{v}, \mu, ||\vec{v}||_1)$ with respect to a multidimensional continued fraction algorithm

$$\vec{v} = M_1 \cdots M_n \, \vec{v}^{(n)}$$

where the $M_i \in SL(3, \mathbb{N})$.

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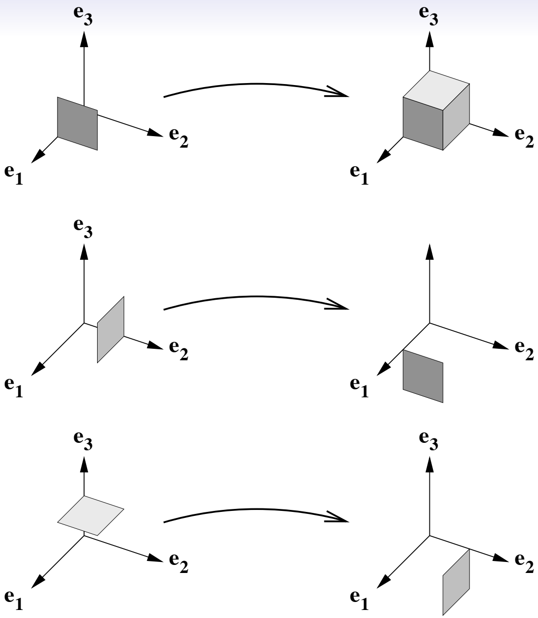
where the $M_i \in SL(3, \mathbb{N})$.

- Take a matrix $M \in SL(3, \mathbb{N})$. We want to find an algorithmic way to go from from

$$\mathcal{P}_{(M\vec{v}, \mu, ||M\vec{v}||_1||)} \text{ to } \mathcal{P}_{(\vec{v}, \mu, ||\vec{v}||_1)}.$$

- We use the fact that

$$\langle \vec{x}, M \, \vec{v} \rangle = \langle {}^t M \, \vec{x}, \vec{v} \rangle$$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Substitutions in word combinatorics

Let σ be a substitution on $\mathcal{A}$.
Example:

$$\sigma(1) = 12,\ \sigma(2) = 13,\ \sigma(3) = 1.$$

The incidence matrix M_σ of σ is defined by

$$M_\sigma = (|\sigma(j)|_i)_{(i,j)\in\mathcal{A}^2}\ ,$$

where $|\sigma(j)|_i$ counts the number of occurrences of the letter i in $\sigma(j)$.

Unimodular substitution

$$\det M_\sigma = \pm 1$$

Generalized substitutions

Abelianisation

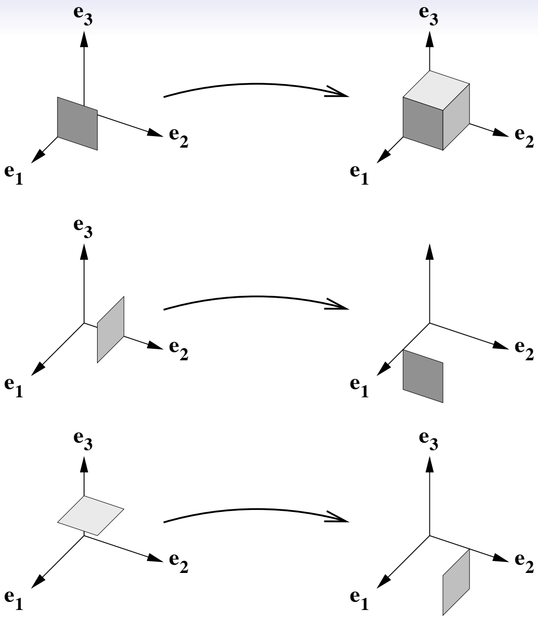
Let d be the cardinality of $\mathcal{A}$. Let $\vec{l} : \mathcal{A}^\star \longrightarrow \mathbb{N}^d$ be the abelinisation map

$$\vec{l}(w) = {}^t(|w|_1, |w|_2, \cdots, |w|_d).$$

Generalized substitutions [P. Arnoux-S. Ito][H. Ei]

Let σ be a unimodular substitution.

$$E_1^*(\sigma)(\vec{x}, i^*) = \sum_{j \in \mathcal{A}} \sum_{P,\, \sigma(j) = PiS} \left(M_\sigma^{-1} \left(\vec{x} + \vec{l}(S) \right), j^* \right).$$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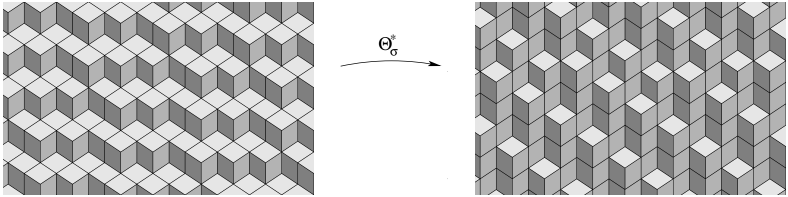
Action on planes and surfaces

Theorem [Arnoux-Ito, Fernique]

Let σ be a unimodular substitution. Let $\vec{v} \in \mathbb{R}^d_+$ be a positive vector. The generalized substitution $E_1^*(\sigma)$ maps without overlaps the stepped plane

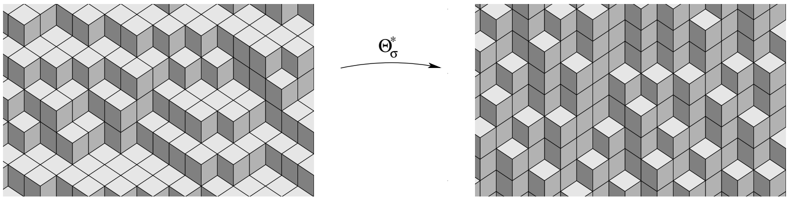
$$\mathfrak{P}(\vec{v}, \mu, ||\vec{v}||_1) \text{ onto } \mathfrak{P}({}^tM_\sigma \vec{v}, \mu, ||{}^tM_\sigma \vec{v}||_1)$$



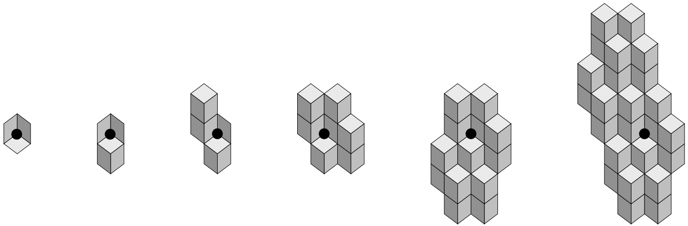
Action on surfaces

Theorem [B.-Fernique]

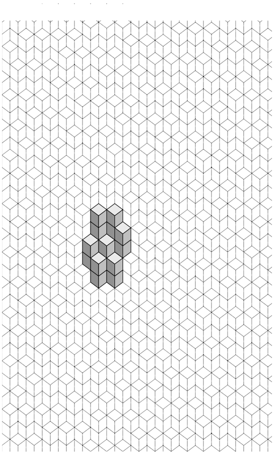
Let σ be a unimodular substitution. The generalized substitution $E_1^*(\sigma)$ maps without overlaps stepped surfaces onto stepped surfaces.



Some iterations



Generation



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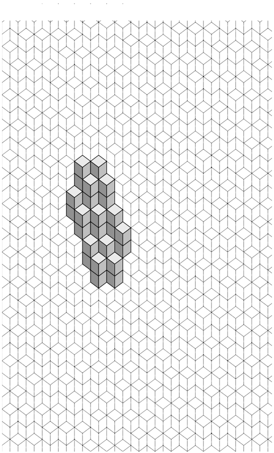

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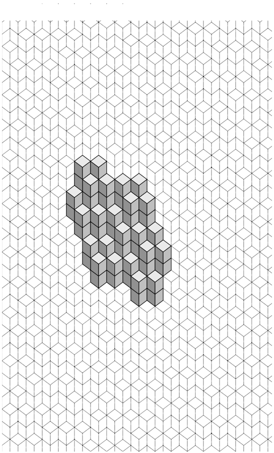

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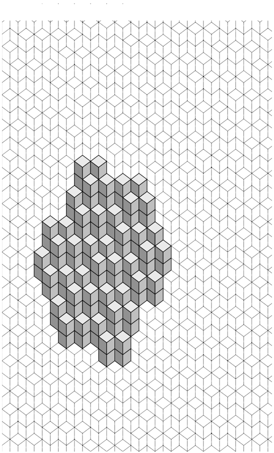

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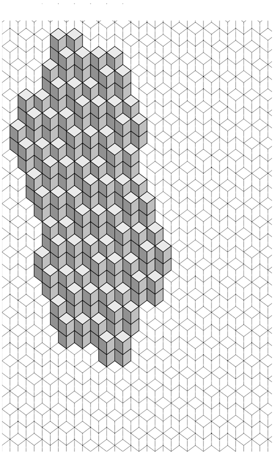

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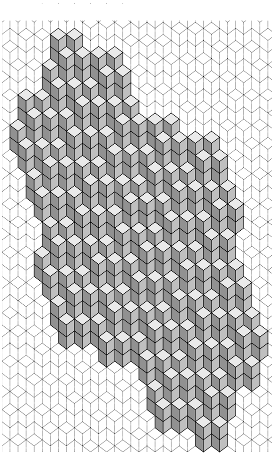

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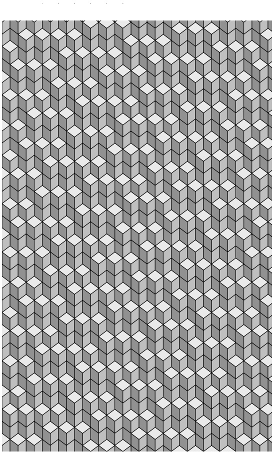
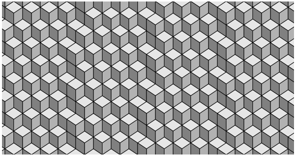


Recognition

One has

$$E_1^*(\sigma \circ \tau) = E_1^*(\tau) \circ E_1^*(\sigma)$$

We can substitute and desubstitute



One gets a recognition algorithm inspired by the Sturmian case based on Brun's algorithm [B.-Fernique, Fernique DGCI'08].

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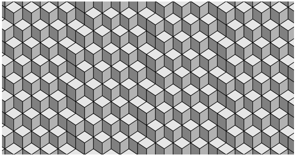

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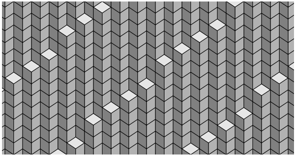

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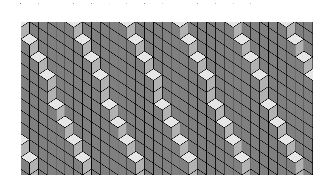

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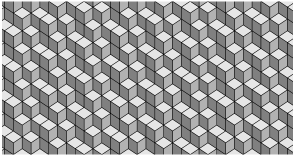

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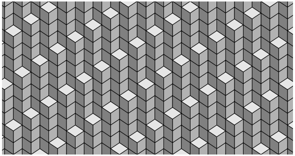

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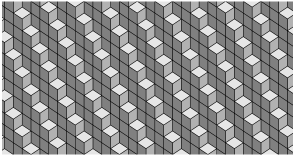

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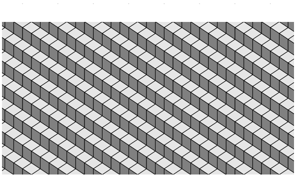

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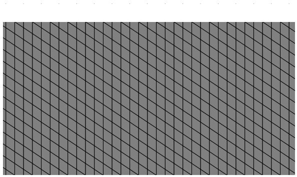

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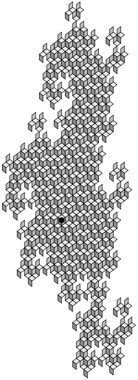
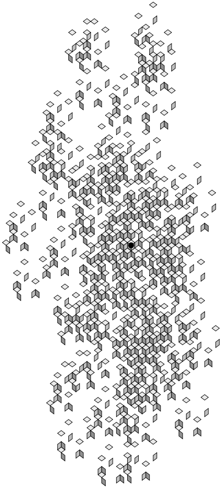
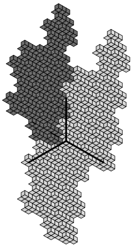
Changing the order of letters





$1 \mapsto 12,\ 2 \mapsto 23,\ 3 \mapsto 123$ $\qquad$ $1 \mapsto 12,\ 2 \mapsto 32,\ 3 \mapsto 231$

Generation of a discrete plane



- By considering the iterates of the unit cube $\mathcal{U}$ under the action of a generalized substitution, are we able to generate the whole discrete plane?
- If yes, what is the shape of these patterns?

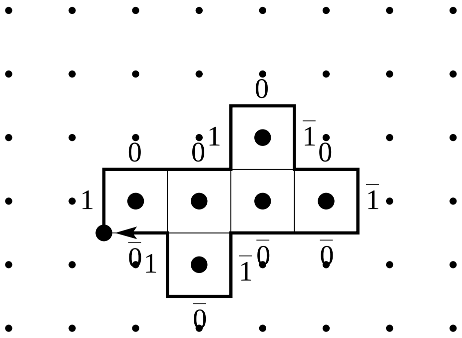
Theorem [B.-Lacasse-Paquin-Provençal] Take any admissible Jacobi-Perron expansion. The boundaries of the patterns

$$E_1^*(\sigma_{(B_1,C_1)}) \ldots E_1^*(\sigma_{(B_n,C_n)})(\mathcal{U})$$

are self-avoiding paths.

Freeman code

The Freeman code gives the segment directions of the contour of a bounded connected shape.



$$w = 1\,0\,0\,1\,0\,\bar{1}\,0\,\bar{1}\,\bar{0}\,\bar{0}\,\bar{1}\,\bar{0}\,1\,\bar{0}$$

Boundary words

A word over the alphabet $\{0, 1, \overline{0}, \overline{1}\}$ is a boundary word if it codes the boundary of a polyomino.

A polyomino is determined by the conjugation class of its boundary word.

Some natural questions...

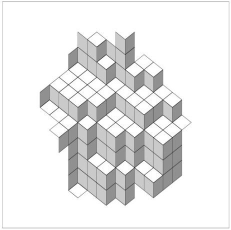
- Can we recognize efficiently whether a word w is a boundary word?
- What can be said on the corresponding polyomino? Can we detect geometric properties like convexity?
- Does it tile the plane?

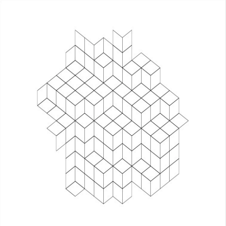
For answers, see the following DGCI's papers [Brlek-Koskas-Provençal'09, Blondin Massé-Brlek-Garon-Labbé'09, Provençal-Lachaud'09, Brlek-Lachaud-Provençal-Reutenauer'08, Brlek-Provençal'06]

From tilings to discrete geometry

$\mathbb{Z}^2 \rightsquigarrow$ Honeycomb/hexagonal and triangular lattice

polyomino $\rightsquigarrow$ polyamond (diamond)





Lozenge tiling model/ Dimers on the honeycomb graph/ Perfect matching of a bipartite planar graph

See [Bodini-Fernique-Rémila, Bodini-Lumbroso DGCI'09]

Back to tilings
Long-range aperiodic order

Discrete planes with irrational normal vector are

- repetitive (uniform recurrence)
- aperiodic

The corresponding tilings are obtained by a cut and project scheme and yield quasicrystals (model sets)

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Multidimensional substitutive tilings $\rightsquigarrow$ Local/matching rules [S. Mozes, C. Goodman-Strauss]

Can we recognize/characterize a given "substitutive" arithmetic discrete plane by local inspection?

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Yes in the Tribonacci case $\sigma : 1 \mapsto 12,\ 2 \mapsto 13,\ 3 \mapsto 1$ [Bressaud-Sablik-Pytheas Fogg'09]

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